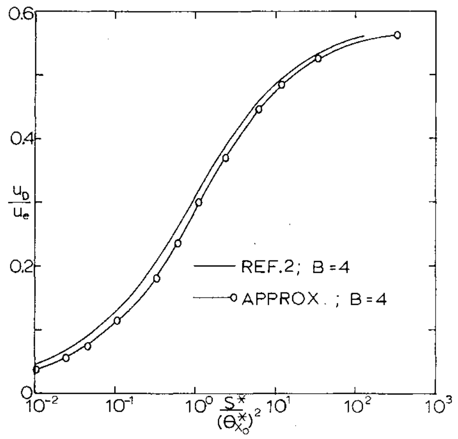

Fig. 4 Dividing streamline velocities for initial blowing profile.

thickness. Integrating from $x_0(S = 0)$ to S gives

$$\theta_{x0}^* = \theta_s^* - \Psi_{ns}^* \quad (9)$$

(subscripts s and x_0 indicate conditions evaluated at S and x_0). Squaring, introducing $\Psi_{ns}^* = 0.5\Psi_{As}^*$ (i.e., for $\epsilon = 3.0$ at point A), and dividing by $(x_0^* + S^*)$, gives after some rearranging

$$S^*/(\theta_{x0}^*)^2 = [\theta_s^*/(x_0^* + s^*)^{1/2} - 0.5\Psi_{As}^*/(x_0^* + S^*)^{1/2}]^{-2} - x_0^*/(\theta x_0^*)^2 \quad (10)$$

For constant-pressure similar inner profiles, the terms in this expression are independent of the shear layer scale. For any assumed inner profile and matching outer profile as specified by the matching point [specified by $\Psi_{As}^*/(x_0^* + S^*)^{1/2}$] the value of $(S^* + x_0^*)^{1/2}/\theta_s^*$ can be related to $\Psi_{As}^*/(x_0^* + S^*)^{1/2}$ using the definition of θ_1. $S^*/(\theta_{x0})^2$ is then evaluated from Eq. (10) for each matching point. The corresponding values of u_D/u_e are given by

$$u_D/u_e = (u_D/u_A)(u_A/u_e) = [F(\beta, 1)/F(\beta, 3)](u_A/u_e)$$

where u_D/u_A is expressed by Eq. (5), and u_A/u_e is given by each specified matching point.

The computed variations in u_D/u_e with $S^*/(\theta_{x0}^*)^2$ are shown in Fig. 3 for the flat-plate inner profile expressed by the Blasius solution, the Pohlhausen polynomial, and a cubic. Excellent agreement is obtained between these results and the exact numerical solution of Ref. 1. The approximate results of Ref. 3 are also shown for comparison. For small values of S, the velocity ratio u_D/u_e corresponds to that for a free layer developing on a nonuniform flow of constant velocity gradient. The solutions for small S and the Blasius and Pohlhausen profiles reduce approximately to the simple relation

$$u_D/u_e = 0.35[S^*/(\theta_{x0}^*)^2]^{1/3} \quad (11)$$

This equation is also plotted on Fig. 3 to show that it represents a reasonable approximation up to $u_D/u_e \approx 0.25$.

The shear layer velocity profiles have also been compared with those of Ref. 1 for an initial Blasius profile. Excellent agreement exists for $u_D/u_e \leqslant 0.40$. For higher values of u_D/u_e, the velocity gradients in the outer profile of the approximate solution become significantly less than those of Ref. 1 because matching becomes progressively more critical. The relative shape of the profiles are essentially correct, however. Thus, good agreement should be obtainable by either improving the matching or by satisfying the momentum equation for both the region of $\Psi > 0$ and $\Psi < 0$. The velocity of the dividing streamline is not sensitive to these differences, however.

Figure 4 shows a comparison of the approximate solution with the exact numerical solution of Ref. 2 for a rather extreme case of the surface blowing profiles of Ref. 6. In this case the present solution results in values of u_D/u_e which are slightly low (about 0.02 below the exact values). This may, in part, be due to the fact that the solution begins with $\beta > 0.50$ giving rise to a poorer asymptotic match at $\epsilon = 3$.

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Kinetic Energy and Angular Momentum about the Variable Center of Mass of a Satellite

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Nomenclature

$\{E\}$	=	unit dyadic
$\mathbf{h}$	=	angular momentum vector
$\{I\}$	=	moment of inertia dyadic
m_j	=	mass of individual body
M	=	total mass of satellite
$\mathbf{R}_j$	=	absolute position vector
$\mathbf{r}_j$	=	relative position vector
t	=	time
T	=	kinetic energy
ω_j	=	angular velocity vector

Subscript

0	=	satellite main body

IT is well known that the motion of a system of particles or rigid bodies can be conveniently separated into a motion of the center of mass of the system and a motion about the center of mass. When these two motions are uncoupled, considerable simplification results. In space applications, this means that the trajectory and attitude motions of a satellite can sometimes be treated separately. A satellite generally consists of a main body and many auxiliary bodies that have motions relative to the main body.† The relative motion of these smaller bodies often may be neglected in studying the orbital motion of the satellite. But frequently it is just the motion of these smaller masses which is responsible for the damping, stability, and instability of the attitude of a satellite. In these cases it is necessary to determine the total kinetic energy and angular momentum of the satellite about its center of mass, taking into consideration the motions of these smaller masses. Now, because of the motions

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† A detailed derivation of the equations of motion of a multiple-part satellite may be found in Refs. 1 and 2.

of these masses, the location of the center of mass of the satellite changes with time. It would be a rather tedius process if one has to determine first the position of the satellite center of mass and then calculate the kinetic energy and angular momentum of the system with respect to this variable center of mass. It is the purpose of this paper to show that, by referring the motion of the auxiliary masses to the center of mass of the satellite main body, the total kinetic energy and angular momentum of the satellite about its center of mass can be found in relatively simple forms without having to determine the location of the variable center of mass explicitly.

Consider a satellite composed of a main body of mass m_0 and N auxiliary bodies of masses m_i ($i = 1, 2, 3, \ldots N$), each of which may translate and rotate with respect to the main body. Let $\mathbf{R}_j$ ($j = 0, 1, 2, \ldots N$) designate the position vectors of the centers of mass of the individual bodies, and

$$\mathbf{r}_j = \mathbf{R}_j - \mathbf{R}_0 \qquad (j = 0, 1, 2, \ldots N) \tag{1}$$

the corresponding position vectors relative to the center of mass of the satellite main body. The position vector $\mathbf{R}$ of the center of mass of the satellite is, by definition,

$$\mathbf{R} = \left(\sum_{j=0}^{N} m_j\mathbf{R}_j\right)\Big/\left(\sum_{j=0}^{N} m_j\right) = \left[\sum_{j=0}^{N} m_j(\mathbf{r}_j + \mathbf{R}_0)\right]\Big/\left(\sum_{j=0}^{N} m_j\right) = \mathbf{R}_0 + \left(\sum_{i=1}^{N} m_i\mathbf{r}\right)\Big/ M \tag{2}$$

where

$$M = \sum_{j=0}^{N} m_j$$

is the total mass of the satellite. With these preliminaries, we shall now proceed to the derivation of the desired results. We shall use the centered dot and multiplication sign between vectors to denote scalar and vector products; when no symbol appears between vectors, a dyadic product is denoted.

1. Kinetic Energy about the Variable Center of Mass of the Satellite

Let us define $\{I_j\}$, ($j = 0, 1, 2, \ldots N$) as the moment of inertia dyadic of each individual body about its own center of mass. The total kinetic energy T of the satellite is

$$T = \left(\frac{1}{2}\right)\sum_{j=0}^{N}\left(\boldsymbol{\omega}_j\cdot\{I_j\}\cdot\boldsymbol{\omega}_j + m_j\frac{d\mathbf{R}_j}{dt}\cdot\frac{d\mathbf{R}_j}{dt}\right) \tag{3}$$

where $\boldsymbol{\omega}_j$ is the angular velocity vector of the jth body.

By repeated use of Eqs. (1) and (2), and after some algebraic manipulations, we can obtain the following final form for the total kinetic energy of the satellite:

$$T = \left(\frac{M}{2}\right)\frac{d\mathbf{R}}{dt}\cdot\frac{d\mathbf{R}}{dt} + \left(\frac{1}{2}\right)\sum_{j=0}^{N}\boldsymbol{\omega}_j\cdot\{I_j\}\cdot\boldsymbol{\omega}_j + \left(\frac{1}{2}\right)\times \sum_{i=1}^{N} m_i\frac{d\mathbf{r}_i}{dt}\cdot\frac{d\mathbf{r}_i}{dt} - \left(\sum_{i=1}^{N} m_i\frac{d\mathbf{r}_i}{dt}\right)\cdot\left(\sum_{i=1}^{N} m_i\frac{d\mathbf{r}_i}{dt}\right)\Big/ 2M \tag{4}$$

Evidently the first term in Eq. (4) is the kinetic energy represented by the center of mass motion of the satellite. The rest of the terms represent the kinetic energy of motion about the satellite center of mass and are of primary interest in studying the attitude motion of a satellite. Notice that we may interpret the last term in Eq. (4) as the correction term to be introduced because the satellite center of mass does not coincide with the main body center of mass.

2. Angular Momentum about the Variable Center of Mass of the Satellite

The total angular momentum h of the satellite about its center of mass can be written as

$$\mathbf{h} = \sum_{j=0}^{N}\left[\{I_j\}\cdot\boldsymbol{\omega}_j + (\mathbf{R}_j - \mathbf{R})\times m_j\frac{d\mathbf{R}_j}{dt}\right] \tag{5}$$

Proceeding as before, it is not difficult to show that Eq. (5) can be rearranged in the following form in which neither $\mathbf{R}$ nor $\mathbf{R}_0$ appears:

$$h = \sum_{j=0}^{N}\{I_j\}\cdot\boldsymbol{\omega}_j + \sum_{i=1}^{N} m_i\mathbf{r}_i\times\frac{d\mathbf{r}}{dt} - \left(\sum_{i=1}^{N} m_i\mathbf{r}_i\right)\times\left(\sum_{i=1}^{N} m_i\frac{d\mathbf{r}_i}{dt}\right)\Big/ M \tag{6}$$

Again the last term in Eq. (6) represents the correction for the fact that the satellite center of mass does not coincide with the main body center of mass.

3. Instantaneous Moment of Inertia Dyadic about the Satellite Center of Mass

It is sometimes of interest to determine the moment of inertia of the satellite about its center of mass, considering the smaller movable masses to be fixed at their instantaneous positions with respect to the satellite main body. For this situation

$$\boldsymbol{\omega}_i = \boldsymbol{\omega}_0 \qquad (i = 1, 2, 3, \ldots N) \tag{7}$$

$$d\mathbf{r}_i/dt = \boldsymbol{\omega}_0\times\mathbf{r}_i \tag{8}$$

and the total angular momentum about the satellite center of mass as given in Eq. (6) becomes

$$\mathbf{h} = \left(\sum_{j=0}^{N}\{I_j\}\right)\cdot\boldsymbol{\omega}_0 + \sum_{i=1}^{N} m_i\mathbf{r}\times(\boldsymbol{\omega}_0\times r_i) - \left(\sum_{i=1}^{N} m_i\mathbf{r}_i\right)\times\left[\boldsymbol{\omega}_0\times\left(\sum_{i=1}^{N} m_i\mathbf{r}_i\right)\right]\Big/ M \tag{9}$$

An alternate expression for the same quantity is

$$\mathbf{h} = \{I\}\cdot\boldsymbol{\omega}_0 \tag{10}$$

By equating Eq. (10) to Eq. (9), the instantaneous moment of inertia dyadic $\{I\}$ about the satellite center of mass can be determined as‡

$$\{I\} = \sum_{j=0}^{N}\{I_j\} + \sum_{i=1}^{N} m_i(\mathbf{r}_i\cdot\mathbf{r}_i\{E\} - \mathbf{r}_i\mathbf{r}_i) - \left[\left(\sum_{i=1}^{N} m_i\cdot\mathbf{r}_i\right)\cdot\left(\sum_{i=1}^{N} m_i\mathbf{r}_i\right)\{E\} - \left(\sum_{i=1}^{N} m_i\mathbf{r}_i\right)\left(\sum_{i=1}^{N} m_i\mathbf{r}_i\right)\right]\Big/ M \tag{11}$$

where $\{E\}$ is the unit dyadic.

4. Application to a Satellite Consisting of a Main Body and a Single Auxiliary body

To illustrate the application of the previous result we shall consider a satellite that consists of a main body and only a single auxiliary body. Equation (4) for the total kinetic energy of the satellite about its center of mass simplifies to

$$T = (\tfrac{1}{2})[\boldsymbol{\omega}_0\cdot\{I_0\}\cdot\boldsymbol{\omega}_0 + \boldsymbol{\omega}_1\cdot\{I_1\}\cdot\boldsymbol{\omega}_1 + m_1(1 - m_1/M)(d\mathbf{r}_1/dt)\cdot(d\mathbf{r}_1/dt)] \tag{12}$$

The angular momentum about the satellite center of mass be-

‡ The author is indebted to Bob Roberson for this explicit expression.

comes, from Eq. (6),

$$\mathbf{h} = \{I_0\}\cdot\boldsymbol{\omega}_0 + \{I_1\}\cdot\boldsymbol{\omega}_1 + m_1(1 - m_1/M)\mathbf{r}_1 \times (d\mathbf{r}_1/dt) \quad (13)$$

Equations (12) and (13) show that we could calculate these kinetic energy and angular momentum as if the main body center of mass were the over-all satellite center of mass provided that we replace the auxiliary body mass m_1 by a reduced mass $m_1(1 - m_1/M)$. As an application of practical interest, we can now apply this result to assess the accuracy of an unsymmetrical yo-yo despin device (see Fig. 1) described in Ref. 3. The length of the yo-yo cord required for reducing the satellite spin to zero is given in Ref. 3 as

$$l_f = (a^2 + I_0/m_1)^{1/2} \quad (14)$$

the derivation being based on the conservation of energy and angular momentum and on the approximation that there is negligible difference between the satellite and system (satellite plus yo-yo) centers of mass. Of course, for the yo-yo to be an efficient despin device, this would be a very good approximation, but from our previous result, we see that correction for this difference can be made with no more difficulty by simply replacing m_1 by the reduced mass $m_1(1 - m_1/M)$ in Eq. (14).§

The instantaneous moment of inertia about the center of mass of the satellite and yo-yo system follows from Eq. (11) as

$$I = I_0 + I_1 + m_1(1 - m_1/M)r_1^2 \quad (15)$$

If we compare this expression with the well-known parallel axes theorem for the moment of inertia, again, the meaning of the reduced mass is clear.

The concept of a reduced mass used in a slightly different meaning is well known in the classical two-body problem (see, for instance, Ref. 4). Notice, however, when there is more than one auxiliary body in a satellite, cross-product terms between the bodies appear in Eqs. (4) and (6), and the simplicity provided by the concept of a reduced mass does not exist.

5. Discussion

The representation of the kinetic energy and angular momentum about the variable satellite center of mass in Eqs. (4) and (6) is valid not only for a satellite vehicle but also for any system of particles and rigid bodies. The relative position and velocity vectors can be those referred to any one of the bodies, whichever one is the most convenient. The advantages of these representations are as follows:

1) It is not necessary to calculate explicitly the location of the variable center of mass.

2) These representations involve only the position and velocities relative to the satellite main body center of mass. This is convenient because a) these relative positions and velocities are usually either those specified, or those which can be expressed in relatively simple analytic forms; and b) the motion of the center of mass of the satellite main body does not appear in the equations.

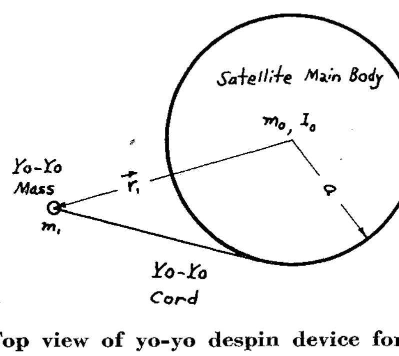

Fig. 1 Top view of yo-yo despin device for a satellite.

§ No correction is made here for the finite moment of inertia of the yo-yo mass about its own center of mass even though the procedure is just as straightforward.

It is expected that the present result should find application to satellite attitude stabilizing systems based on the principle of using small masses to execute prescribed motions with respect to the satellite main body.

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Linear Feedback Solutions for Minimum Effort Interception, Rendezvous, and Soft Landing

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Statement of General Problem

THERE are two moving bodies in vacuo, one the pursuing body (subscript P) and the other a target body (subscript T). The equations of motion for the two bodies are

$$\dot{\mathbf{v}}_P = \mathbf{a}_P + \mathbf{u} \qquad \dot{\mathbf{r}}_P = \mathbf{v}_P \quad (1)$$

$$\dot{\mathbf{v}}_T = \mathbf{a}_T \qquad \dot{\mathbf{r}}_T = \mathbf{a}_T \quad (2)$$

where $\mathbf{v}_P$, $\mathbf{v}_T$ = velocity vectors, $\mathbf{r}_P$, $\mathbf{r}_T$ = position vectors, $\mathbf{a}_T$ = acceleration vectors due to external forces, $\mathbf{u}$ = acceleration vector due to control force, pursuing body, $(\dot{\ }) = d(\ )/dt$. The relative motion is described by

$$\dot{\mathbf{v}} = \mathbf{a} + \mathbf{u} \qquad \dot{\mathbf{r}} = \mathbf{v} \quad (3)$$

where $\mathbf{v} = \mathbf{v}_P - \mathbf{v}_T$, $\mathbf{r} = \mathbf{r}_P - \mathbf{r}_T$, $\mathbf{a} = \mathbf{a}_P - \mathbf{a}_T$. The problem is to find $\mathbf{u}(t)$ to minimize

$$J = \frac{1}{2}\,[c_v\mathbf{v}\cdot\mathbf{v} + c_r\mathbf{r}\cdot\mathbf{r}]_{t=T} + \frac{1}{2}\int_{t_0}^{T} \mathbf{u}\cdot\mathbf{u}dt \quad (4)$$

where T = terminal time, t_0 = initial time, c_v, c_r are scalar constants. The Hamiltonian of the problem is therefore

$$H = (\mathbf{u}\cdot\mathbf{u})/2 + \boldsymbol{\lambda}_v\cdot(\mathbf{a} + \mathbf{u}) + \boldsymbol{\lambda}_r\cdot\mathbf{v} \quad (5)$$

and the influence vectors are determined by

$$\dot{\boldsymbol{\lambda}}_v = -\partial H/\partial\mathbf{v} = -\boldsymbol{\lambda}_r \quad (6)$$

$$\dot{\boldsymbol{\lambda}}_r = -\partial H/\partial\mathbf{r} = 0 \quad (7)$$

where it was assumed that $\mathbf{a}$ was constant. If both pursuer and target are being acted upon by the same (constant) gravitational force per unit mass, then $\mathbf{a} = 0$. If the target is a fixed point, as in the soft-landing problem, then $\mathbf{a}_T = 0$ and $\mathbf{a} = \mathbf{a}_P = \mathbf{g}$ is the gravitational force per unit mass. Even though $\mathbf{g}$ is never truly constant, the approximation of treating it as constant is quite good for short flight paths.

The optimality condition is simply

$$\partial H/\partial\mathbf{u} = \mathbf{u} + \boldsymbol{\lambda}_v = 0 \rightarrow \mathbf{u} = -\boldsymbol{\lambda}_v \quad (8)$$

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